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On Efficiency and Pareto Optimality of Competitive Programs in Closed Multisector Models*

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I. INTRODUCTION

One of the well-known paradoxes of infinity is the possibility that a competitive program is inefficient, such inefficiency being linked to over-accumulation of capital. Recognition of the serious implications of this fact has led to attempts to derive conditions that can isolate completely the set of efficient competitive programs. However, these conditions seem to depend on rather specific properties of the technology, and even among the simpler economic models there are basic qualitative differences in the criteria for characterizing completely the set of efficient competitive programs. Nevertheless, in order to gain a proper understanding of the

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role of prices in guiding resource allocation over time in a decentralized or a centrally planned economy, it is essential that we have easily applicable criteria identifying the efficient competitive programs, at least for the more important models of intertemporal resource allocation. For the neoclassical model a fundamental result has recently been established by Cass [2], and his characterization has been extended to some ultisector open models [3] (see also [18]). No parallel study has yet been undertaken for *closed* multisector models, i.e., models in which all commodities are producible. The first purpose of this paper is to point out that for a large class of closed multisector models, a program is efficient if and only if it satisfies the intertemporal profit maximization condition relative to a nonnull sequence of *nonnegative*[1] price vectors and the transversality condition that the values of inputs at these competitive prices goes to zero. It is immediately seen (in Sect. IV) that this characterization is quite different from the one obtained by Cass. The Cass criterion does not apply to our framework, just as our condition need not necessarily be satisfied by an efficient program in an open model. The class of models considered in this paper includes, in particular, those of Dorfman, Samuelson, and Solow [5] and McKenzie [15], in which the technology permits *output substitution*, as well as those of Samuelson and Solow [24], Morishima [19], and Nikaido [20], in which there is the possibility of *input substitution*. In view of the extensive use of such closed models in theoretical and empirical literature on growth and planning (see, for example, [4, 22, 26]), and the fact that our substitution assumptions are perhaps the most commonplace ones in economic theory, as reflected in the "usual" shapes of isoquants and production possibility curves, a unified and systematic presentation providing a complete characterization of efficient programs in these models will hopefully be of some interest.

It is known from [17] that a sufficient condition for efficiency is the existence of a sequence of *strictly positive* competitive prices relative to which the transversality condition holds. However, even in finite dimensions and with output substitution in the technology an efficient program need not have strictly positive competitive prices, and for infinite programs even when the prices happen to be strictly positive, the transversality condition does not necessarily hold in open models (recall the "golden rule" examples!). The important fact that with substitution possibilities in closed models, in which there is a strictly positive vector of von Neumann stocks, *nonnegative* competitive prices together with the

[1] An m-vector $x = (x^i)$ is *nonnegative* (written $x \geqq 0$) if $x^i \geqq 0$ for all i. It is semipositive (written $x > 0$) if $x \geqq 0$ and $x \neq 0$. It is *strictly positive* (written $x \gg 0$) if $x^i > 0$ for all i. A sequence $p = (p_t)$ of m-vectors is *nonnull* if $p_t \neq 0$ for at least one t.

transversality condition are equivalent to efficiency, has not appeared in the literature, although the adequacy of the transversality condition in signalling capital overaccumulation has often been discussed (see [25, p. 273]).

Assuming the strict positivity of competitive prices, Kurz [12, p. 281] and Kurz and Starrett [13, pp. 575, 576] were able to show that the Malinvaud prices associated with an efficient program must necessarily satisfy the transversality condition when (a) the efficient program is "locally contractable" or (b) "productive." However, one can show that an efficient program in a *closed* model can never satisfy the local contractability assumption and, as Kurz himself recognized, the condition of productivity is too strong and is not implied by the substitution conditions that we shall consider. In our proof of the necessity of transversality in the closed model, we show the existence of the system of competitive prices supporting the efficient program such that the present value of any feasible program is finite and is maximized at the given efficient program. Note that present value maximization is stronger than the properties usually obtained in the more general framework of Malinvaud [17].

A second purpose of the paper is to apply our result to characterize Pareto optimal programs in a model with overlapping generations and to relate the problem of Pareto optimal *distribution* over time to the problem of efficient *allocation* of resources in this framework. Actually, following Samuelson [23], it is conventional to examine the distribution question in a "productionless" economy—where the agents have given endowments for exchange. We introduce production in a simple way, and in the extended model, our main result (under differentiability assumptions) is that a program is long-run Pareto optimal if and only if it is short-run Pareto optimal and efficient. Thus, roughly speaking, the problem of a "proper" distribution of goods is essentially a short-run feature and the only long-run problem—the only paradox of infinity—is one of inefficiency or capital overaccumulation. This proposition was first proved by Bose [1] for a neoclassical framework. Our exercise supplements his work and indicates that the proposition is valid, for a more general class of models.

II. Efficiency in Technologies with Output Substitution

IIa. *The Model*

The framework chosen here is the familiar closed model of production (see, for example, [20] or [19, Chap. VI], for detailed interpretation). Consider an economy in which there are m producible goods. The technology does not change over time, and is described by a set $\mathscr{T}$ in the nonnegative orthant of R^{2m}—a pair (x, y) is in $\mathscr{T}$ if and only if it is possible to get the output vector y in period $(t + 1)$ by using the input vector x in period t. The following assumptions on $\mathscr{T}$ are maintained *throughout this paper*:

(A.1) *$\mathscr{T}$ is a closed convex cone in the nonnegative orthant of R^{2m} (continuity, convexity, and constant returns to scale).*

(A.2) *$(0, y) \in \mathscr{T}$ implies $y = 0$ (impossibility of free production).*

(A.3) *There is $(\bar{x}, \bar{y}) \in \mathscr{T}$ with $\bar{y} \gg 0$ (producibility).*

(A.4) *$(x, y) \in \mathscr{T}$ and $x' \geqslant x$, $0 \leqslant y' \leqslant y$ imply $(x', y') \in \mathscr{T}$ (free disposal).*

As usual, for any $(x, y) \in \mathscr{T}$ with $x > 0$, let $\lambda(x, y) = \max\{\lambda: y \geqslant \lambda x\}$. It is known (see [9, p. 338]) that under (A.1) through (A.4), there are $(\hat{x}, \hat{y}) \in \mathscr{T}$, $\hat{\lambda} > 0$ ($\hat{\lambda}$ is finite), and a price vector $\hat{p} > 0$ such that

$$\hat{\lambda} = \lambda(\hat{x}, \hat{y}), \quad \hat{y} = \hat{\lambda}\hat{x}, \quad \hat{\lambda} \geqslant \lambda(x, y) \quad \text{for all } (x, y) \in \mathscr{T} \text{ with } x > 0,$$
$$\hat{p}y \leqslant \hat{\lambda}\hat{p}x \quad \text{for all } (x, y) \in \mathscr{T}. \tag{2.1}$$

We follow the usual convention of referring to $\hat{p}$ as a *von Neumann price vector*, $\hat{x}$ as a vector of *von Neumann stocks*, and $\hat{\lambda}$ as the von Neumann *growth factor*. In what follows, *we shall assume without loss of generality, that $\hat{\lambda} = 1$, in order to simplify notation.* Given any $\mathscr{T}'$ satisfying the above assumptions, one simply takes the corresponding *present value technology* $\mathscr{T} = \{(x, y): (x, \hat{\lambda}y) \in \mathscr{T}'\}$. $\mathscr{T}$ has the same structure as $\mathscr{T}'$, and obviously has a maximal Von Neumann growth factor equal to one which is is achievable at any vector of input proportions at which $\hat{\lambda}$ is achievable in $\mathscr{T}'$. The interested reader is referred to the paper by Winter [27, p. 68–9] for details, and is invited to verify that the assumptions made below are not in conflict with this convention. *Keeping in mind that $\hat{\lambda} = 1$, the next assumption can be stated simply as*

(A.5) *There is some $\hat{x} \gg 0$ such that $(\hat{x}, \hat{x}) \in \mathscr{T}$.*

In other words, we assume that there is a strictly positive vector of von Neumann stocks. Next, we define a *feasible production program* from $\mathbf{x}$ as a sequence $(x, y) = (x_t, y_{t+1})$ such that

$$x_0 = \mathbf{x}, \quad x_t \leqslant y_t \quad \text{for all } t \geqslant 1,$$
$$(x_t, y_{t+1}) \in \mathscr{T} \quad \text{for all } t \geqslant 0. \tag{2.2}$$

The consumption program $c = (c_t)$ generated by (x, y) is defined as:

$$c_t = y_t - x_t \ (\geqslant 0) \qquad \text{for all } t \geqslant 1. \tag{2.3}$$

We refer to (x, y, c) as a *feasible program* from $\mathbf{x}$, it being understood that (x, y) is a production program and c is the corresponding consumption program. A feasible program (x^*, y^*, c^*) from $\mathbf{x}$ is efficient if there is no other feasible program (x, y, c) from $\mathbf{x}$ such that $c_t \geqslant c_t^*$ for all t and $c_t > c_t^*$ for some t. A feasible program (x^*, y^*, c^*) from $\mathbf{x}$ is *competitive* if there is a nonzero sequence (p_t^*) of nonnegative price vectors such that for all $t \geqslant 0$ one has

$$0 = p_{t+1}^* y_{t+1}^* - p_t^* x_t^* \geqslant p_{t+1}^* y - p_t^* x \qquad \text{for all } (x, y) \text{ in } \mathcal{T}. \tag{2.4}$$

In other words, the *intertemporal profit maximization* condition (2.4) is satisfied for all t. A competitive program (x^*, y^*, c^*) satisfies the *transversality condition* if $p_t^* x_t^*$ goes to zero as t goes to infinity.

IIb. *Technologies with Output Substitution*

We now introduce the concept of output substitution. Essentially it is required that if it is possible to produce y from x with $y^i > 0$, then for any commodity j $(\neq i)$ it is also possible to produce more than y^j of it from x with a suitable reduction of y^i, keeping the outputs of all other commodities unchanged. More formally, we have

(A.6) *Suppose that* $(x, y) \in \mathcal{T}$ *with* $y^i > 0$ *for some* i. *Given any* $j \neq i$ *and any* δ_i *satisfying* $0 < \delta_i \leqslant y^i$, *there exists* $\delta_j > 0$ *such that* $(x, y') \in \mathcal{T}$ *where* $y'^i = y^i - \delta_i$, $y'^j = y^j + \delta_j$, *and* $y'^k = y^k$ *for all* $k \neq i, j$.

Note that δ_j in general depends on δ_i as well as on the (x, y) under consideration. In Fig. 1, technologies (a) and (b) satisfy output substitution, while (c) does not. Using convexity, one can show that if $(x, y) \in \mathcal{T}$, $y \geqslant \mathbf{y} \geqslant 0$, and $y^i > \mathbf{y}^i$ for some i, then by (A.5) there exists y' with $(x, y') \in \mathcal{T}$ and $y' \gg \mathbf{y}$.[2] Two examples of technologies with output substitution will be given.

EXAMPLE 2.1. The polyhedral $\mathcal{T}$ defined as $\mathcal{T} = \{(x, y): Az \leqslant x, Bz \geqslant y, z \geqslant 0\}$ where A is an $n \times n$ strictly positive matrix and B is the $n \times n$ identity matrix, satisfies (A.6). In general, however, if $\mathcal{T}$ *is a polyhedral convex cone output substitution may not be possible.*

[2] Since the argument is used repeatedly in our proofs, we spell it out completely. Choose $0 < \delta_i < y^i - \mathbf{y}^i$. For each $j \neq i$, there exists $\delta_j > 0$ such that $(x, y - \delta_i\omega_i + \delta_j\omega_j) \in \mathcal{T}$, where ω_i is a vector with a one in the ith place, zeros elsewhere. By convexity, $(x, y') = (1/(m-1)) \sum_{j\neq 1} (x, y - \delta_i\omega_i + \delta_j\omega_j) \in \mathcal{T}$, and $y' \gg \mathbf{y}$.

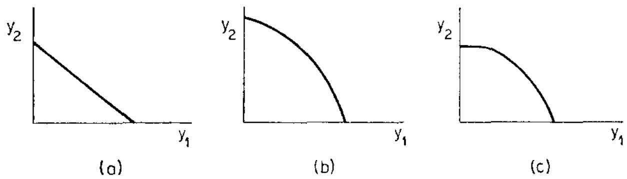

FIGURE 1

EXAMPLE 2.2. Let F be a nonnegative real-valued function on the nonnegative orthant of R^{2m-1} such that it is continuously differentiable, concave, and homogeneous of degree one. Define

$$\mathscr{T} = \{(x, y): 0 \leqslant y^m \leqslant F(x^1,..., x^m, y^1,..., y^{m-1})\}.$$

This is the well-known neoclassical transformation process of Dorfman, Samuelson, and Solow [5]. $F(\ )$ gives the maximum value of the output of the mth good given the values of its arguments. It will be assumed that $\partial F/\partial x^i > 0$ for $i = 1,..., m$ and $\partial F/\partial y^i < 0$ for $i = 1, 2,..., m - 1$, and verification of the properties listed above is easy.

While the requirement that the technology satisfies (A.6) may be strong, it is clear that (A.6) does not guarantee that for an efficient consumption program,[3] the associated prices (p_t) are strictly positive. The following theorem settles the question of relating efficient to competitive programs, when the technology satisfies (A.1) through (A.6):

THEOREM 2.1. *Under* (A.1) *through* (A.6) *a feasible program* (x^*, y^*, c^*) *from* $\mathbf{x} \gg 0$ *is efficient if and only if there exists a nonnull sequence* (p_t^*) *of nonnegative price vectors satisfying for all* $t = 1, 2,...,$

$$0 = p_{t-1}^* y_{t+1}^* - p_t^* x_t^* \geqslant p_{t+1}^* y - p_t^* x \qquad \text{for all } (x, y) \in \mathscr{T} \quad (2.5)$$

and

$$\lim_{t \to \infty} p_t^* x_t^* = 0. \quad (2.6)$$

Proof. (Sufficiency). Suppose that (x^*, y^*, c^*) is a feasible program from $\mathbf{x}$ such that there exists a nonnull sequence (p_t^*) of nonnegative price vectors satisfying Eqs. (2.5) and (2.6). We have to prove that

[3] Consider the example of Arrow (see [8, p. 88, Footnote 52]) where $\mathscr{T} = \{(x, y) \geqslant 0: (y^1)^2 + (y^2)^2 \leqslant [\min(x^1, x^2)]^2\}$. However the (productive) efficient point $y^* = (1, 0)$ produced from $x^* = (1, 1)$ can be supported only by the price system $p_{y1} = 1$ and $p_{y2} = 0$.

(x^*, y^*, c^*) is efficient. To this effect we start by showing that for any feasible program (x, y, c) from $\mathbf{x}$ one has

$$\sum_{t=1}^{\infty} p_t{}^* c_t \leqslant \sum_{t=1}^{\infty} p_t{}^* c_t{}^* = p_0{}^* \mathbf{x}. \tag{2.7}$$

Feasibility of (x, y, c) from $\mathbf{x}$ and Eq. (2.5) guarantee that for any $T \geqslant 1$,

$$S_T = \sum_{t=1}^{T} p_t{}^* c_t \leqslant p_0{}^* \mathbf{x}. \tag{2.8}$$

Nonnegativity of p_t and c_t implies that S_T is a monotonically nondecreasing sequence which by (2.8) is bounded above. Hence, $\lim_{T\to\infty} S_T$ exists and clearly

$$\lim_{T\to\infty} S_T = \sum_{t=1}^{\infty} p_t{}^* c_t \leqslant p_0{}^* \mathbf{x}. \tag{2.9}$$

For the particular program (x^*, y^*, c^*) under consideration, one has $\sum_{t=1}^{T} p_t{}^* c_t{}^* = p_0{}^* \mathbf{x} - p_T{}^* x_T{}^*$. Using Eq. (2.6) and taking limits one has Eq. (2.7).

Next, an important property of the competitive prices $(p_t{}^*)$ satisfying Eq. (2.5) is noted:

$$p_t{}^* = 0 \textit{ implies } p_{t+1}^* = 0 \quad \textit{for all } t \geqslant 0; \textit{ hence } p_0{}^* > 0;\ p_1{}^* > 0. \tag{2.10}$$

This conclusion does *not* depend on the transversality condition Eq. (2.6). To establish Eq. (2.10) note that if (a) $p_T{}^* = 0$ for some $T \geqslant 1$, and $p_{T+1}^* > 0$, Eq. (2.5) implies that $p_{T+1}^* y - p_T{}^* x \leqslant 0$ for all $(x, y) \in \mathscr{T}$. By (A.3), $p_{T+1}^* \bar{y} > 0 = p_T{}^* \bar{x}$, a contradiction. If (b) $p_0{}^* = 0$ and $p_1{}^* > 0$, we have again by Eq. (2.5), $p_1{}^* y - p_0{}^* \mathbf{x} \leqslant 0$ for all $(x, y) \in \mathscr{T}$. By (A.3) there is some $\beta > 0$ such that $(\mathbf{x}, \beta\bar{y}) \in \mathscr{T}$. Hence, $0 < p_1{}^* \beta \bar{y} \leqslant p_0{}^* \mathbf{x} = 0$, a contradiction. By (a) and (b), $p_t{}^* = 0$ implies $p_{t+1}^* = 0$ for all $t \geqslant 0$. Since the sequence $(p_t{}^*)$ is not null, $p_0{}^* > 0$. Finally, since $p_0{}^* \mathbf{x} > 0$, we have by Eq. (2.5), $p_1{}^* y_1{}^* = p_0{}^* \mathbf{x} > 0$ which means that $p_1{}^* > 0$, completing the proof of Eq. (2.10).

We now come to the proof of the result. Suppose that (x^*, y^*, c^*) is *not* efficient. This means that there is a feasible program $(\tilde{x}, \tilde{y}, \tilde{c})$ from $\mathbf{x}$ such that $\tilde{c}_t \geqslant c_t{}^*$ for all $t \geqslant 1$ and $\tilde{c}_t > c_t{}^*$ for some t, say $t = t' \geqslant 1$. Either $p_{t'}^* = 0$ (Case I) or $p_{t'}^* > 0$ (Case II). We examine each case in turn for a contradiction.

Case I. Consider the last period τ, $1 \leqslant \tau < t'$, for which $p_\tau{}^* > 0$. Since $\tilde{y}_{t'} \geqslant \tilde{c}_{t'} > c_{t'}^* \geqslant 0$, (A.2) implies $\tilde{y}_{t'-1} \geqslant \tilde{x}_{t'-1} > 0$. Repeating this process, we finally get $\tilde{x}_\tau > 0$. Construct a feasible program $(\tilde{x}'', \tilde{y}'', \tilde{c}'')$

from $\mathbf{x}$ with $\tilde{c}_t'' = \tilde{c}_t$ for $t < \tau$, $\tilde{c}_\tau'' = \tilde{c}_\tau + \tilde{x}_\tau$, and $\tilde{c}_t'' = 0$ for $t > \tau$. By (A.6), one can construct a feasible program $(\tilde{x}', \tilde{y}', \tilde{c}')$ from $\mathbf{x}$ such that $\tilde{c}_t' = \tilde{c}_t''$ for $t \neq \tau$ and $\tilde{c}_\tau' \gg c_\tau^*$. Then using (2.10), $\sum_{t=1}^{\infty} p_t^* \tilde{c}_t' = \sum_{t=1}^{\tau} p_t^* c_t' > \sum_{t=1}^{\tau} p_t^* c_t^* = \sum_{t=1}^{\infty} p_t^* c_t^*$, a contradiction of (2.7).

Case II. Using (A.5), one can construct a program $(\tilde{x}', \tilde{y}', \tilde{c}')$ from $\mathbf{x}$ such that $\tilde{c}_t' = \tilde{c}_t$ for $t \neq t'$ and $\tilde{c}_{t'} \gg c_{t'}^*$. Since $p_t > 0$, this implies $\sum_{t=1}^{\infty} p_t^* \tilde{c}_t' > \sum_{t=1}^{\infty} p_t^* c_t^*$, contradicting (2.7). Thus, the sufficiency part of the theorem is proved.

(Necessity). An important consequence of (A.5) and (A.6) is that

the von Neumann price vector $\hat{p}$ *is strictly positive.* (2.11)

Since $\hat{p}$ is semipositive, for some j one has $\hat{p}^j > 0$. Suppose that for some i, $\hat{p}^i = 0$. By (A.5) we have $(\hat{x}, \hat{x}) \in \mathscr{T}$ where $\hat{x} \gg 0$. By applying (A.6) we have $(\hat{x}, y) \in \mathscr{T}$ where $y^i > \hat{x}^i$, $y^j < \hat{x}^j$ and $y^k = \hat{x}^k$ for $k \neq i, j$. But $\hat{p}y - \hat{p}\hat{x} = \hat{p}^i y^i - \hat{p}^i \hat{x}^i > 0$, contradicting the definition of $\hat{p}$ (see Eq. (2.1) keeping in mind that $\hat{\lambda} = 1$). This establishes Eq. (2.11).

Since by (A.5), $(\hat{x}, \hat{x}) \in \mathscr{T}$ and $\hat{x} \gg 0$, the following useful property is obvious:

There exists $\hat{\sigma} > 0$ *such that* $(\hat{x}, \hat{\sigma}\omega) \in \mathscr{T}$ *where* $\omega = (1, \ldots, 1) \in R^m$. (2.12)

Define $G = \{c = (c_t)\colon c_t = y_t - x_t$ for all $t \geqslant 1$, $x_0 = \mathbf{x}$, $(x_t, y_{t+1}) \in \mathscr{T}$, for all $t \geqslant 0\}$. Clearly, G contains all *feasible* consumption programs $c = (c_t)$, which satisfy these properties and the additional requirement that $c_t \geqslant 0$ for all t. By Eq. (2.1) for any *feasible* consumption program $c = (c_t)$ one has for all $T \geqslant 0$

$$\sum_{t=1}^{T} \hat{p} c_t \leqslant \hat{p}\mathbf{x} - \hat{p} x_T \leqslant \hat{p}\mathbf{x}. \tag{2.13}$$

Since $\hat{p} \gg 0$ (by Eq. 2.11), for any feasible consumption program $c = (c_t)$

$$\| c \| = \sum_{t=1}^{\infty} | c_t | \leqslant \alpha \hat{p}\mathbf{x}, \qquad \text{where} \quad | c_t | = \sum_{i=1}^{m} | c_t^i |, \tag{2.14}$$

where $\alpha > 0$ is determined by $\hat{p}$.

Let $\mathscr{X}$ be the linear space of all sequences $c = (c_t)$ such that $\| c \|$ is finite. An element p of $\mathscr{X}^*$, the set of all continuous linear functionals on $\mathscr{X}$ can be represented[4] as a sequence $p = (p_t)$ such that $\| p \|_* = \sup_t | p_t |_*$

[4] See [21, p. 64] or [6, p. 289].

is finite, where $| p_t |_* = \max_i | p_t^i |$. Let $\mathscr{F} = (G \cap \mathscr{X}) - \mathscr{X}^+$, where $\mathscr{X}^+$ is the set of all nonnegative sequences in $\mathscr{X}$. $\mathscr{F}$ is easily seen to be a convex and closed (under pointwise convergence) subset of $\mathscr{X}$.[5]

Let $\gamma > 0$ be such that $\gamma \hat{x} \leqslant \mathbf{x}$. The program $c^\tau = (0,..., 0, \hat{\sigma}\gamma\omega, 0,...)$ generated by pure accumulation at the von Neumann growth rate until $\tau - 1$, followed by the activity given in Eq. (2.12) to yield consumption in period τ is feasible. Hence c^τ is in $\mathscr{F}$ for all $\tau \geqslant 1$. Clearly $c^0 = 0$ is also in $\mathscr{F}$.

Consider any c' in $\mathscr{X}$ satisfying $\| c' \| < \hat{\sigma}\gamma$, and define $\theta_t = | c_t' | / \hat{\sigma}\gamma$ for $t \geqslant 1$ and $\theta_0 = 1 - \sum_{t=1}^\infty \theta_t \geqslant 0$. Then $c' \leqslant \sum_{\tau=0}^\infty \theta_\tau c^\tau = c''$ and c'' is contained in $\mathscr{F}$ by convexity and closedness under pointwise convergence. Hence $c' \in \mathscr{F}$ and we have proved that $\mathscr{F}$ has an interior point.

Consider the given efficient program (x^*, y^*, c^*). It is easy to check that c^* is in the boundary of $\mathscr{F}$. Hence by a separation argument there is[6] a nonzero continuous linear functional $p^* = (p_t^*)$ on $\mathscr{X}$ satisfying

$$\sum_{t=1}^{\infty} p_t^* c_t^* \geqslant \sum_{t=1}^{\infty} p_t^* c_t \qquad \text{for all } c = (c_t) \text{ in } \mathscr{F}. \tag{2.15}$$

Since $\mathscr{F}$ contains all nonpositive sequences in $\mathscr{X}$, p_t^* must be nonnegative for each t. The proof will be completed by a demonstration that (2.15) implies the competitive condition Eq. (2.5).

For $\tau \geqslant 1$, define $c_t' = c_t'' = c_t^*$ for all $t \neq \tau, \tau + 1$; $c_\tau' = c_\tau^* - x_\tau$; $c_\tau'' = c_\tau^* + x_\tau^*$; $c_{\tau+1}' = c_{\tau+1}^* + y_{\tau+1}$; $c_{\tau+1}'' = c_{\tau+1}^* - y_{\tau+1}^*$. One can verify that c' results from augmenting production in τ by $(x_\tau, y_{\tau+1})$ in $\mathscr{T}$, and c'' results from reducing production to zero in τ. Hence, c' and c'' are in $\mathscr{F}$, implying from (2.15) that

$$p_{\tau+1}^* y_{\tau+1} - p_\tau^* x_\tau \leqslant 0 \qquad \text{for all } (x_\tau, y_{\tau+1}) \in \mathscr{T},\ \tau \geqslant 1, \tag{2.16}$$

$$p_{\tau+1}^* y_{\tau+1}^* - p_\tau^* x_\tau^* = 0 \qquad \text{for} \quad \tau \geqslant 1. \tag{2.17}$$

Next consider $\bar{c}$ defined by $\bar{c}_1 = y_1^*$, $\bar{c}_t = 0$ for $t > 1$. Clearly $\bar{c}$ is in $\mathscr{F}$, and $\sum_{t=1}^\infty p_t^* \bar{c}_t = p_1^* y_1^* \leqslant \sum_{t=1}^\infty p_t^* c_t^*$. On the other hand, summing the conditions (2.17),

$$0 = \sum_{t=1}^{T-1} (p_{t+1}^* y_{t+1}^* - p_t^* x_t^*) = \sum_{t=1}^{T} p_t^* c_t^* + p_T^* x_T^* - p_1^* y_1^*.$$

[5] One can easily adapt the arguments in [14, Lemma II, p. 45].

[6] See [10, Theorem 14.2].

Hence,

$$\lim_{T\to\infty} p_T{}^* x_T{}^* = p_1{}^* y_1{}^* - \lim_{T\to\infty} \sum_{t=1}^{T} p_j{}^* c_t{}^* = p_1{}^* y_1{}^* - \sum_{t=1}^{\infty} p_j{}^* c_t{}^* \leqslant 0. \tag{2.18}$$

Since $p_T{}^* x_T{}^* \geqslant 0$, this implies the transversality condition (2.6).

Note that $p_1{}^* > 0$; otherwise, the argument following Eq. (2.10) would imply that the sequence p^* is null for a contradiction. The supposition that there exists $(\mathbf{x}, y) \in \mathscr{T}$ with $p_1{}^* y > p_1{}^* y_1{}^*$ implies the program $\mathbf{c}'$ in $\mathscr{F}$ with $\mathbf{c}_1{}' = y$, $\mathbf{c}_t{}' = 0$ for $t > 1$ would contradict (2.15). Thus,

$$z^* \equiv p_1{}^* y_1{}^* \geqslant p_1{}^* y \qquad \text{for all } (\mathbf{x}, y) \in \mathscr{T}. \tag{2.19}$$

Define the convex set K as

$$K = \{(x, z) \in R^m \times R\colon x \geqslant 0, z \leqslant p_1{}^* y \text{ for any } y \text{ such that } (x, y) \in \mathscr{T}\}.$$

Notice that Eq. (2.19) implies that $(\mathbf{x}, z^*)$ is a boundary point of K, which clearly has interior points. Hence, there is some *nonzero* (p, λ') in $R^m \times R$ such that

$$p\mathbf{x} + \lambda' z^* \geqslant px + \lambda' z \qquad \text{for all } (x, z) \text{ in } K. \tag{2.20}$$

Note that (a) $\lambda' < 0$ is impossible (choose $(\mathbf{x}, z)$ with $z < z^*$ on the right-hand side of Eq. (2.20); (b) $p = 0$ or $p^i > 0$ is impossible (choose $(\beta'\mathbf{x}, \beta' y_1{}^*)$ in $\mathscr{T}$ for a sufficiently large β' to contradict Eq. (2.20)); (c) $\lambda' = 0$ is impossible since $p\mathbf{x} \geqslant px$ for all $x \geqslant 0$ and $-p > 0$, $\mathbf{x} \gg 0$ will also contradict Eq. (2.20)). Hence, define $p_0{}^* = -(1/\lambda')\,p$. Clearly $p_0{}^* > 0$ and for any $(x, y) \in \mathscr{T}$ one has $p_1{}^* y - p_0{}^* x = p_1{}^* y + (px/\lambda') = (1/\lambda') \times [\lambda' p_1{}^* y + px] \leqslant (1/\lambda')[\lambda' z^* + p\mathbf{x}] = z^* + (p/\lambda')\,\mathbf{x} = p_1{}^* y_1{}^* - p_0{}^*\mathbf{x} = 0$. This completes the proof of Eq. (2.4) as well as the necessity part of the theorem. Q.E.D.

Remark 1. It should be emphasized that the condition of *present value maximization* (Eq. 2.15) is a result of independent interest and does *not* follow from the well-known alternative approaches leading to the existence of Malinvaud prices— in a closed model such prices can be shown to exist under assumptions (A.1) through (A.4) see, e.g., [17] or [21].

Remark 2. Note that the sufficiency half of the theorem does *not* depend on (A.5), the strict positivity of von Neumann stocks. On the other hand, the *necessity* part of the theorem remains valid if (A.6) is replaced by

(A.7) $\hat{p} \gg 0$, i.e., *there is a strictly positive von Neumann price vector.*

It has been shown that the properties (A.5) and (A.7) follow from indecomposability and some other assumptions on $\mathscr{T}$ (see e.g., [20, p. 205;

19, p. 180]. Actually, (A.7) holds whenever 0 is a maximal point of the closure $\bar{Z}$ of the convex cone $Z = \{y - x: (x, y) \in \mathscr{T}\}$. (See [20, pp. 35–36].)

In view of the remarks above, and for a convenient organization of the proofs of the following sections (in which the assumption of output substitution is never made), it is useful to have the following precise statement to refer to:

THEOREM 2.2. *Under assumptions* (A.1) *through* (A.5) *and* (A.7), *if a feasible program* (x^*, y^*, c^*) *from* $\mathbf{x} \gg 0$ *is efficient, then there is a nonnull sequence* (p_t^*) *of nonnegative price vectors such that for all* $t \geqslant 0$

$$0 = p_{t+1}^* y_{t+1}^* - p_t^* x_t^* \geqslant p_{t+1}^* y - p_t^* x \qquad \textit{for all } (x, y) \in \mathscr{T} \tag{2.21}$$

and

$$\lim_{t \to \infty} p_t^* x_t^* = 0 \tag{2.22}$$

III. SOME FURTHER RESULTS

IIIa. *Input Substitution*

In this section we present some results for models in which output substitution does not hold. In what follows, however, we do have to restrict (see Example 3.2 below) our analysis to efficient programs that are "interior" in the sense of using strictly positive input vectors at each date. Formally, a feasible probram (x, y, c) from $\mathbf{x}$ is an *interior* program if $x_t \gg 0$ for all t. While the restriction to interior programs is somewhat ad hoc, it is weaker than the assumption of Cass [2] requiring a strictly positive lower bound on input levels. It is easy to check that *if an interior program is competitive, the associated price vectors* (p_t) *must satisfy* $p_t > 0$ *for all* t,[7] *although it is not necessary* that $p_t \gg 0$.[8]

First, instead of the assumption of output substitution we consider the following assumption of input substitution.

(A.6′) *Suppose that* $(x, y) \in \mathscr{T}$ *and* $x^i > 0$ *for some* i. *Given any* $j \neq i$ *and* $\delta_j > 0$, *there exists* δ_i *satisfying* $0 < \delta_i < x^i$ *such that* $(x', y) \in \mathscr{T}$ *where* $x'^i = x^i - \delta_i$, $x'^j = x^j + \delta_j$ *and* $x'^k = x^k$ *for all* $k \neq i, j$.

The following theorem is easily proved:

[7] Let t be the first period for which $p_t = 0$. From Eq. (2.10), $t > 0$. Then $0 < p_{t-1} x_{t-1} = p_t y_t$, for a contradiction.

[8] The example of Arrow (Footnote 2) can be easily modified to show this. See, for example, [12, Diagram 1 and discussion, p. 289].

THEOREM 3.1. *Under* (A.1) *through* (A.5), *and* (A.6′), *and interior program* (x^*, y^*, c^*) *from* $\mathbf{x} \gg 0$ *is efficient if and only if there exists a nonnull sequence* (p_t^*) *of nonnegative price vectors such that for all* $t = 0, 1, 2, \ldots,$

$$0 = p_{t+1}^* y_{t+1}^* - p_t^* x_t^* \geqslant p_{t+1}^* y - p_t^* x \qquad \textit{for all } (x, y) \in \mathscr{T} \quad (3.1)$$

and

$$\lim_{t \to \infty} p_t^* x_t^* = 0. \quad (3.2)$$

Proof. (Sufficiency). In view of Malinvaud's theorem it is enough to show, by using (A.6′), that the sequence (p_t^*) satisfying (3.1) and (3.2) also satisfies $p_t^* \gg 0$ for all t. Suppose for some t and i, $p_t^i = 0$. Since $p_t > 0$, $p_t^j > 0$ for some $j \neq i$ and by (A.6′) there is $(\bar{x}, y_{t+1}^*) \in \mathscr{T}$ where $\bar{x}^j < x_t^{*j}$, $\bar{x}^i > x_t^{*i}$ and $\bar{x}^k = x_t^{*k}$ for $k \neq i, j$. Then $p_{t+1}^* y_{t+1}^* - p_t^* \bar{x} > 0$, contradicting Eq. (3.1).

(Necessity). Use Theorem 2.2, since (A.6′) and $\hat{x} \gg 0$ imply $\hat{p} \gg 0$ (using exactly the argument leading to the strict positivity of competitive prices in the sufficiency part).

In the discussion above, the restriction to interior programs seems unavoidable. It is instructive to look at the following:

EXAMPLE 3.1. Let $\mathscr{T} = \{(x, y) : y^1, y^2 \leqslant ((x^1)^{1/2} + (x^2)^{1/2})^2/4,\ x \geqslant 0,\ y \geqslant 0\}$, with $\mathbf{x} = (1, 1)$. This is a technology satisfying (A.5′) but not (A.5). The program $y_1 = (1, 1)$, $c_1 = (1, 0)$, $c_t = (0, 0)$ for all $t \geqslant 2$ is seen to be inefficient, but satisfies (3.1) and (3.2) with respect to prices $p_0 = p_1 = (1, 0)$ and $p_t = (0, 0)$ for $t \geqslant 2$.

In connection with the application of our Theorem 3.1 to the "sausage machine" technologies of Samuelson and Solow, it should be mentioned that the theorem can also be proved if (A.5′) is replaced by the condition of *primitivity* appearing in the related literature (see [19, p. 179]). This condition requires that for any $\mathbf{x} > 0$ there is a finite sequence $(x_t, y_{t+1}) \in \mathscr{T}$ such that $x_0 = \mathbf{x}$, $y_T \gg 0$.

IIIb. *The Polyhedral Case: A Counterexample*

The substitution conditions discussed above are typically *not* satisfied when the technology is a polyhedral convex cone (i.e., generated by a finite set of basic activities, (see, for example, [11, p. 79]). The sufficiency half of Theorem 2.1 ceases to be valid, as can be seen from the simplest examples. It is natural to inquire whether by using the polyhedral structure one can sharpen the necessity half of that theorem to derive a sequence of strictly positive competititve prices supporting an efficient program. Together with Malinvaud's sufficiency theorem, this would then provide us with a complete characterization in terms of strictly positive competitive prices directly analogous to a standard result applicable to models with a *finite* number of commodities and a technology that is a polyhedral convex cone (see [7, pp. 306–308] or [20, pp. 186–187]). The remarks in [11, p. 111, Footnote 1] seem to suggest that this is indeed possible.

We now give an example of an efficient program in a simple polyhedral model with three goods such that there is no sequence of strictly positive prices (p_t^*) that are competitive. Let $\mathscr{T} = \{(x, y) \geqslant 0,\ y \geqslant Cx\}$, with

$$C = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}, \qquad C^{-1} = \begin{bmatrix} \frac{2}{3} & -\frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 \\ -\frac{1}{3} & \frac{1}{6} & \frac{1}{2} \end{bmatrix}.$$

Consider the program $\mathbf{x} = x_t = (1, 1, 1)$, $y_t = (3, 3, 3)$, $c_t = (2, 2, 2)$ for $t \geqslant 1$. This program corresponds to production and consumption in von Neumann proportions, and is clearly efficient. Prices must satisfy the difference equation equation $p_{t+1}^* = C^{-1} p_t^*$, which has the general solution

$$p_t^* = c_1(\tfrac{1}{3})^t(1, 1, 0) + c_2(\tfrac{1}{2})^t(0, 1, -1) + c_3(1, -1, 0).$$

The only solution allowing $p_t^* \geqslant 0$ for all t is $c_1 > 0$, $c_2 = c_3 = 0$, implying $p_t^{*3} = 0$, as was to be demonstrated. Note that $p_t^* = (\tfrac{1}{3})^t, (\tfrac{1}{3})^t, 0)$ satisfies the transversality condition as well as the competitive conditions.

IIIc. *Technological Change*

One can dispense with the assumption that the technology of the economy does not change over time. We shall sketch a possible generalization, referring the interested reader to the analysis of McFadden [14] for further details. Suppose that (the present value) technology $\mathscr{T}_t$ at date t $(= 0, 1, \ldots)$ satisfies (A.1) through (A.4). The sufficiency part of Theorem 2.1 can be established by following the same arguments as above if $\mathscr{T}_t$ satisfies (A.6) for all t. To obtain an extension of the necessity part, the assumptions can be recast in the following manner. Let $\mathscr{T}^*$ be the smallest closed convex cone containing all the (present value) technologies $\mathscr{T}_t$. The necessity part is obtained if, in addition to (A.1) through (A.4) holding for each $\mathscr{T}_t$, one has

(A.8) *$\mathscr{T}^*$ has a von Neumann growth rate equal to one*; $(0, y)$ *in $\mathscr{T}^*$ implies $y = 0$; and $\mathscr{T}^*$ contains no sequence (x_n, y_n) with $y_n - x_n$ having a nonnegative nonzero limit point.*

(A.8′) *There exists a positive scalar* σ *such that for each t commodity vectors* $b^1,\ldots, b^t$ *can be found with* (ω, b^1) *in* $\mathscr{T}_0$, (b^1, b^2) *in* $\mathscr{T}_1 \cdots (b^{t-1}, b^t)$ *in* $\mathscr{T}_{t-1}$ *and* $b^j \geqslant \sigma\omega$ *where* $\omega = (1,\ldots, 1)$.

It follows from (A.8) that there is $\hat{p} \gg 0$ such that $\hat{p}(y - x) = 0$ for all (x, y) in $\mathscr{T}^*$. If technological change is neutral or biased towards balanced growth and there is a strictly positive von Neumann ray contained in all $\mathscr{T}_t$, then (A.8′) holds.

IV. Open Versus Closed Models

The results obtained in Sections II, IIIa, and IIIc supplement that of Majumdar [16] in which the technology need not permit such substitution and yet efficiency can be completely characterized in terms of intertemporal profit maximization and transversality conditions (see [16]). However, this characterization is quite different from that of Cass [2]. For the neoclassical model, or for a closed one-good model in which the technology is described by a production function f that is strictly concave in C^2, the Cass criterion is applicable to interior programs satisfying

$$\underline{\underline{x}} \geqslant \underline{x}_t \geqslant \bar{x} > 0 \qquad \text{for all } t, \tag{4.1}$$

where x_t is the input (per capita capital in the neoclassical case) at date t. The competitive prices (p_t) associated with a program are defined by $p_t = 1/\pi_t$ with $\pi_t = \prod_{s=0}^{t-1} f'(x_s)$, $\pi_0 = 1$. Inefficiency of a program (x, y, c) is equivalent to finiteness of $\lim_{T\to\infty} \sum_{t=0}^{T} \pi_t$. In view of the bounds (4.1), we easily see that *inefficiency of* (x, y, c) *is equivalent to finiteness of*

$$\lim_{T\to\infty} \sum_{t=0}^{T} (1/p_t x_t). \tag{4.2}$$

It is easy to figure out the relation between (4.2) and the transversality condition and the essential difference between the two criteria. It would, of course, be convenient if one could attribute the qualitative difference *entirely* to the presence or absence of a nonproducible commodity (like labor that sets an upper bound on all feasible per capita variables in the neoclassical model). That this is, strictly speaking, not the case will be clear from the following special example.

Example 4.1. The Leontief model of Gale [7, pp. 300–301]. The technology is $\mathscr{T} = \{(l, x, y) \geqslant 0,\ Ay \leqslant x,\ ay \leqslant l\}$, where l is the input of "labor" (the nonproducible good that is not consumed), A is an $n \times n$

positive matrix and a is a strictly positive vector giving the input requirements of producible goods and labor respectively. The condition for a program (l^*, x^*, y^*, c^*) to be competitive is that there exist nonnegative sequences (p_t^*), (w_t^*) such that

$$0 = p_{t+1}^* y_{t+1}^* - p_t^* x_t^* - w_t^* l_t^* \geqslant p_{t+1}^* y_{t+1} - p_t^* x_t - w_j^* l_t$$

for $(l_t, x_t, y_{t+1}) \in \mathscr{T}$. This requires

$$p_{t+1}^* \leqslant p_t^* A + w_t^* a$$

and $(p_{t+1}^* - p_t^* A - w_t^* a) y_{t+1}^* = 0$. If $y_t^* \gg 0$, so that equality holds, the general solution of this difference equation is

$$p_t^* = p_1^* A^{t-1} + \sum_{j=1}^{t-1} w_{t-j}^* a A^{j-1}.$$

An efficient program satisfies $\sum_{j=1}^{t-1} A^j c_j = Ay_1 - A^t y_t$ with $A^t y_t \to 0$. Clearly taking $w_t^* = 0$ provides prices for which $p_t^* y_t \to 0$ is a criterion for efficiency.[9]

V. An Application to the Problem of Characterizing Pareto Optimality

This section applies some of the previous results to provide a complete characterization of Pareto optimal distributions when consumers of overlapping generations are introduced in the model. The questions of *efficient allocation* of resources in production and *Pareto optimal distributions* of goods among consumers are usually treated distinctly in the literature, the former following the lead of Malinvaud, and the latter following Samuelson's pure exchange model. We consider both the questions in a multisector model with production. A study of Pareto optimality in which for each period there is a utility function u_t defined on aggregate consumption c_t and Pareto comparisons among alternative programs are

[9] The result follows directly from the fact that a program (x^*, y^*, c^*) is efficient if $\sum_{t=1}^{\infty} A^{t-1} c_t^* = y_1^*$. Proof of sufficiency as well as the basic steps in necessity is exactly the same as in [16]. To note the only difference, observe that if an efficient program has $\sum_{t=1}^{\infty} A^{t-1} c_t^* = y_1^* - \delta$ with $\delta > 0$, then $y_1' = y_1^*$, $y_s' = \sum_{t=s}^{\infty} A^{t-s} c_t^*$, $x_{s-1}' = Ay_s'$ for $s \geqslant 2$, $c_1' = c_1^* + \delta$, $c_s' = c_s^*$ for $s \geqslant 2$ constitute a feasible program. To check that the labor constraint is satisfied, $y_s' = \sum_{t=s}^{\infty} A^{t-s} c_t^* \leqslant y_s^*$ so that $ay_s' \leqslant ay_s^* \leqslant l_s$. It follows that (c_t') dominates (c_t^*).

made with the utility sequences $u_t(c_t)$ is far more obviously related to the analysis of efficiency (see [17]). In fact, if there is just one good and the utility functions u_t are monotonically increasing, then the two problems are structurally identical since a consumption program (c_t) is efficient if and only if the corresponding utility sequence $u_t(c_t)$ is Pareto optimal. With many goods, a result parallel to Theorem 5.1 also has been worked out by us to characterize long-run Pareto optimality in that framework. The distributional "paradoxes of infinity" involved in the Samuelsonian case of overlapping generations in which Pareto comparisons are made with lifetime utilities have been the object of much discussion and, therefore, seems to be the more natural framework for a detailed study. The technology is that of the closed Dorfman–Samuelson–Solow model in which substitution conditions hold, and we indulge in differentiability assumptions to keep the exposition simple and to get the sharpest result. A more general analysis of Pareto optimality in infinite horizon economies is undoubtedly of importance, and will be the subject of a forthcoming paper.

Va. *The Model with Overlapping Generations*

To keep the notation simple, consumers are assumed to live for 2 periods. Those born at the beginning of period t and dying at the end of period $t + 1$ constitute the tth generation. We have verified[10] that *what follows is valid if the consumers are assumed to live for a finite number* >2 *periods*, and no change in the strategy of the proof is necessary (excepting introduction of some more involved notation). The preferences of all consumers of a particular generation are alike, and are conveniently represented by a real-valued concave strongly monotonic continuous utility function $U_t({}^1c, {}^2c)$ on $(R^{2m})_+$; where ${}^jc = ({}^jc^i)$ is the consumption vector of the tth generation in the jth period of its lifetime ($j = 1, 2$). The utility function U_t is continuously differentiable in the interior of $(R^{2m})_+$. The technology used in this section satisfies the assumptions listed in Example 3.2, and is described by the DOSSO transformation locus F, so that we have

$$\mathscr{T} = \{(x, y): 0 \leqslant y^m \leqslant F(y^1,..., y^{m-1}; x), \qquad x \geqslant 0\}. \tag{5.1}$$

We assume that $\mathscr{T}$ *satisfies* (A.5).

The initial stocks $\mathbf{x} \gg 0$ and the consumption of the "old" people in period 1, denoted by ${}^2c_1$ are assumed to be given. A feasible program

[10] See the earlier version which appeared as Discussion Paper No. 85 circulated by the Department of Economics, Cornell University.

(u, x, y, c) from $(\mathbf{x}, \mathbf{c})$ consists of nonnegative sequences $-(u_t, (x_t), (y_{t+1}), (c_{t+1})-$ satisfying

$$\begin{aligned} x_0 &= \mathbf{x}, \qquad {}^2c_1 = \mathbf{c}; \\ y_{t+1} &= c_{t+1} + x_{t+1}, \qquad (x_t, y_{t+1}) \in \mathscr{T} \qquad \text{for all } t \geqslant 0; \qquad (5.2) \\ c_{t+1} &= {}^2c_{t+1} + {}^1c_{t+1} \qquad \text{for all } t \geqslant 0; \end{aligned}$$

and $u_t = U_t({}^1c_t, {}^2c_{t+1})$ is the sequence of utilities of different generations from the proposed schemes of allocation and distribution.

We restrict our attention to *regular interior programs* (u, x, y, c) satisfying $x_t \gg 0$ and ${}^jc_t \gg 0$ for all t. A feasible program (u^*, x^*, y^*, c^*) is *short-run Pareto-optimal* if there is no other feasible program (u, x, y, c) such that

$$[x_{t+1}, u_1, \ldots, u_t, ({}^1c_{t+1})] > [x_{t+1}^*, u_1^*, \ldots, u_t^*, ({}^1c_{t+1}^*)] \qquad (5.3)$$

for some finite $t \geqslant 1$. It is *long-run Pareto-optimal* is there is no other feasible program (u, x, y, c) with $u_t \geqslant u_t^*$ for all $t \geqslant 1$, strict inequality being valid for some t.

It is immediate that a long-run Pareto-optimal program is necessarily short-run Pareto-optimal. However, the converse is not true—a program such that every finite segment is short-run Pareto-optimal need not be long-run Pareto-optimal. This justifies the above distinction. Our objective is to identify long-run Pareto-optimality with efficient, short-run Pareto-optimal programs, and thus to establish the link between efficiency and Pareto-optimality.

THEOREM 5.1. *A regular interior program* (u^*, x^*, y^*, c^*) *from* $(\mathbf{x}, \mathbf{c})$ *is long-run Pareto-optimal if and only if* (a) *it is short-run Pareto-optimal and* (b) *efficient.*

Proof. Necessity being obvious, let us go directly to the nontrivial sufficiency part. The first step is to note that by using Kuhn–Tucker theorem, if a regular interior program (u^*, x^*, y^*, c^*) from $(\mathbf{x}, \mathbf{c})$ is short-run Pareto-optimal, there is a sequence (q_t^*, p_t^*) of price vectors with $q_t^* > 0$ (in R) and $p_t^* \gg 0$ (in R^m) such that

$$\begin{aligned} &\text{(i)} \quad q_t^* u_t^* - p_t^* \, {}^1c_t^* - p_{t+1}^* \, {}^2c_{t+1}^* \geqslant q_t^* U_t({}^1c, {}^2c) - p_t^* \, {}^1c - p_{t+1}^* \, {}^2c \\ &\qquad\qquad \text{for all } ({}^1c, {}^2c) \geqslant 0 \text{ and all } t \geqslant 1, \\ &\text{(ii)} \quad p_t^* y_t^* - p_{t-1}^* x_{t-1}^* \geqslant p_t^* y - p_{t-1}^* x \\ &\qquad\qquad \text{for all } (x, y) \in \mathscr{T} \text{ and } t \geqslant 1. \end{aligned} \qquad (5.4)$$

Actually, by exploiting the differentiability assumption, the prices in Eq. (5.4) can simply be defined as:

$$p_0^{*i} = \partial F/\partial x_0^{\,i}\ (i = 1,\dots, m);$$

$$p_1^{*i} = -\partial F/\partial y_1^{\,i}\ (i = 1,\dots, m-1); \qquad p_1^{*m} = 1$$

for

$$t \geqslant 2, \qquad p_t^{*m} = 1/\pi_t\,, \qquad p_t^{*i} = -\partial F/\partial y_t^{\,i}/\pi_t\ (i = 1,\dots, m-1), \tag{5.5}$$

where

$$\pi_t = \prod_{s=1}^{t-1} \partial F/\partial x_s^{\,m};$$

$$q_0{}^* = q_1{}^* = 1 \qquad \text{and} \qquad q_t{}^* = (\partial U_t/\partial^1 c_t^{\,m})/\pi_t \qquad \text{for } t \geqslant 2.$$

It is understood that all the derivatives in Eq. (5.5) are computed at (u^*, x^*, y^*, c^*). The computational details leading to Eqs. (5.4) and (5.5) are tedious but straightforward, and are omitted.[11]

[11] The problem is to maximize $F(x_{T+1}^{*1} + c_{T+1}^1,\dots, x_{T+1}^{m-1} + c_{T+1}^{m-1}; x_T)$ subject to

$$U_t({}^1c_t\,, {}^2c_{t+1}) \geqq u_t{}^*,$$

$${}^1c_{T+1} \geqq {}^1c_{T+1}^*,$$

among the set of feasible programs from **x**. Set up the Lagrangean

$$L(\cdot) = F(x_{T+1}^{*1} + c_{T+1}^1,\dots, x_{T+1}^{*m-1} + c_{T+1}^{m-1}\,; x_T) + \sum_{t=1}^{T} \lambda_t[U_t({}^1c_t\,, {}^2c_{t+1}) - u_t{}^*]$$

$$+ \sum_{i=1}^{m} \mu_{T+1}^i({}^1c_{T+1}^i - {}^1c_{T+1}^{*i}).$$

To check the constraint qualification, we construct a feasible program from **x**, which satisfies all the constraints with strict inequality. Let

$$K = \max_{i,t}\,[x_t^{*i}, y_{t+1}^{*i}], \qquad i = 1,\dots, m,$$

$$k = \min_{i,t}\,[x_t^{*i}, y_{t+1}^{*i}], \qquad t = 0,\dots, T.$$

By assumption that the program (u^*, x^*, y^*, c^*) is regular interior, we know that $k > 0$.

For $t \ni 0 \leqslant t \leqslant T$, since from x_t^*, output y_{t+1}^* is producible, so the output $\hat{y}_{t+1}$ given by $[\hat{y}_{t+1}^1,\dots, \hat{y}_{t+1}^{m-1}] = (y_{t+1}^{*1},\dots, y_{t+1}^{*m-1})$ and $0 < \hat{y}_{t+1}^m < y_{t+1}^{*m}$ is also producible [by the derivative conditions on $F(\cdot)$]. Also $\tilde{y}_{t+1} \gg 0$ is producible from $\tilde{x}_t$, given by $[\tilde{x}_t^1,\dots, \tilde{x}_t^{m-1}] = [x_t^{*1},\dots, x_t^{*m-1}]$ and $\tilde{x}_t^m = x_t^{*m} - \epsilon_t$ where $0 < \epsilon_t < k/2$. By taking a suitable convex combination $(0 < \lambda_t < 1)$, from $\bar{x}_t = [x_t^{*1},\dots, x_t^{*m-1}, x_t^{*m} - \lambda_t\epsilon_t]$, we

Suppose that a regular interior program (u^*, x^*, y^*, c^*) is short-run optimal and efficient, but is not long-run optimal. There there is some program $(\tilde{u}, \tilde{x}, \tilde{y}, \tilde{c})$, such that $\tilde{u}_t \geqslant u_t^*$ for all $t \geqslant 1$, strict inequality being valid for at least one $t = \bar{t}$. Let $\tilde{u}_{\bar{t}} - u_{\bar{t}}^* = \gamma' > 0$. Then, by Eq. (5.4) we have for all $T \geqslant \bar{t}$

$$q_{\bar{t}}^*\gamma' \leqslant \sum_{t=1}^{T} q_t^*(\tilde{u}_t - u_t^*) \leqslant p_{T+1}^*(x_{T+1}^* - \tilde{x}_{T+1}) + p_{T+1}^*({}^1c_{T+1}^* - {}^1\tilde{c}_{T+1})$$

$$\leqslant p_{T+1}^* x_{T+1}^* + p_{T+1}^* c_{T+1}^* . \tag{5.6}$$

Since $\sum_{t=1}^{\infty} p_t^* c_t^* \leqslant p_0^* \mathbf{x}$ (by Eq. (2.7)), there is $\bar{T}$ such that for all $T \geqslant \bar{T}$, $p_{T+1}^* c_{T+1}^* \leqslant q_{\bar{t}}^* \gamma'/4$. Note that due to the differentiability assumptions, the prices (p_t^*) defined in Eq. (5.5) are the unique competitive prices associated with the program (u^*, x^*, y^*, c^*). Hence, by Theorem 2.1, the transversality condition Eq. (2.6) must necessarily be satisfied at these prices, i.e., $\lim_{t\to\infty} p_t^* x_t^* = 0$. Hence, there is T^* such that

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can produce $\bar{\bar{y}}_{t+1}$ given by $[\bar{\bar{y}}'_{t+1}, \ldots, \bar{\bar{y}}_{t+1}^{m-1}] = [y_{t+1}^{*1} + \eta_{t+1}^1, \ y_{t+1}^{*m-1} + \eta_{t+1}^{m-1}]$, where $\eta_{t+1}^i > 0$ for all i and $\bar{\bar{y}}_{t+1}^m > 0$. Let $\delta_t = \lambda_t \epsilon_t$ and $\delta = \min_t \delta_t$.

Since F_{x^m} is continuous on $[k/2, K]$, so there exists $M > 0$, such that $F_{x^m} \leqslant M$, for $k \geqslant (x^i, y^i) \geqslant k/2$ for all i.

Now, let $\mu_0 = 0$, $\mu_t = \min[\delta/(2M)^{T+1-t}, 1]$, $t = 1, \ldots, T+1$, and construct the required program (u, x, y, c) as follows. $x_0 = \mathbf{x}$; and for $t = 0, \ldots, T$,

$$y_{t+1} = [\bar{\bar{y}}_{t+1}^1, \ldots, \bar{\bar{y}}_{t+1}^{m-1}, y_{t+1}^{*m} - \mu_{t+1}],$$

$$x_{t+1} = [x_{t+1}^{*1}, \ldots, x_{t+1}^{*m-1}, x_{t+1}^{*m} - 2\mu_{t+1}],$$

$$c_{t+1} = [c_{t+1}^{*1} + \eta_{t+1}^1, \ldots, c_{t+1}^{*m-1} + \eta_{t+1}^{m-1}, c_{t+1}^{*m} + \mu_{t+1}],$$

$${}^1c_{t+1} = [{}^1c_{t+1}^{*1} + \eta_{t+1}^1, \ldots, {}^1c_{t+1}^{*m-1} + \eta_{t+1}^{m-1}, {}^1c_{t+1}^{*m} + \mu_{t+1}],$$

$${}^2c_{t+1} = [{}^2c_{t+1}^{*1}, \ldots, {}^2c_{t+1}^{*m}].$$

To check feasibility note that

(a) $(x_{t+1}, y_{t+1}, {}^1c_{t+1}, {}^2c_{t+1}) \gg 0$, $t = 0, \ldots, T$,

(b) $y_{t+1} = x_{t+1} + c_{t+1}$; $c_{t+1} = {}^1c_{t+1} + {}^2c_{t+1}$; $t = 0, \ldots, T$,

(c) $(x_t, y_{t+1}) \in \mathscr{T}$ $t = 0, \ldots, T$.

To see this, note that since $\mu_t \leqslant \delta \leqslant \delta_t = \lambda_t \epsilon_t$, so $F(y_{t+1}^1, \ldots, y_{t+1}^{m-1}, x_t^m) > 0$, and $F(y_{t+1}^1, \ldots, y_{t+1}^{m-1}, x_t^m) - F(y_{t+1}^{*1}, \ldots, y_{t+1}^{*m-1}, x_t^m) = \bar{F}_x{}^t m(-2\mu_t) > -\delta(2M)\mu_t/(2M)^{T+1-t} \geqslant -\mu_{t+1}$.

$\therefore F(y_{t+1}^1, \ldots, y_{t+1}^{m-1}, x_t^m) > F(y_{t+1}^{*1}, \ldots, y_{t+1}^{*m-1}, x_t^{*m}) - \mu_{t+1} = y_{t+1}^{*m} - \mu_{t+1} = y_{t+1}$. To check that the constraints are satisfied with inequality note that (i) by the derivative conditions on $U_t(\cdot)$, $t = 1, \ldots, T$, $U_t({}^1c_t, {}^2c_{t+1}) > u_t^*$, and (ii) ${}^1c_{T+1} \gg {}^1c_{T+1}^*$.

$p^*_{t+1}x^*_{t+1} \leqslant q_{\bar{t}}{}^*\gamma'/4$ for all $t \geqslant T$. Thus, for $T \geqslant \max(\bar{t}, T^*, \bar{T})$ we have $q_{\bar{t}}{}^*\gamma' \leqslant q_{\bar{t}}{}^*\gamma'/2$, a contradiction. Q.E.D.

Remark. The assumptions on F rule out the cases in which the derivatives becomes infinite at zero values of some input or output. In order to allow for such situations, we can require the derivative conditions to hold only at $(y^1,..., y^{m-1}, x) \gg 0$ and appeal to Section III.

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